

Between Chebyshev and Cantelli

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Abstract

A family of exact upper bounds interpolating between Chebyshev's and Cantelli's is presented.

Let X be any zero-mean unit-variance random variable (r.v.): $\mathbf{E} X = 0$ and $\mathbf{Var} X = 1$. (Obviously, for any non-degenerate r.v. Y with a finite second moment, its standardization $\frac{Y - \mathbf{E} Y}{\sqrt{\mathbf{Var} Y}}$ is a zero-mean unit-variance r.v.) Take any $b \in (0, \infty)$. Chebyshev's inequality states that

$$\mathbf{P}(|X| \geq b) \leq \frac{1}{b^2}. \quad (1)$$

Cantelli's bound on the probabilities of one-sided deviations of X from 0 is obviously smaller:

$$\mathbf{P}(X \geq b) \leq \frac{1}{1 + b^2}. \quad (2)$$

Moreover, Cantelli's bound is exact, as it is attained when X takes on values $-1/b$ and b with probabilities $\frac{b^2}{1+b^2}$ and $\frac{1}{1+b^2}$, respectively. Chebyshev's bound is also exact, but only for $b \geq 1$: indeed, let X take on values $-b$, 0 , and b with probabilities $\frac{1}{2b^2}$, $1 - \frac{1}{b^2}$, and $\frac{1}{2b^2}$, respectively. The obviously modified Chebyshev's bound given by the inequality

$$\mathbf{P}(|X| \geq b) \leq 1 \wedge \frac{1}{b^2} \quad (3)$$

is exact for all $b \in (0, \infty)$; indeed, for $b \in (0, 1)$ let X take on each of the values ± 1 with probability $\frac{1}{2}$. Clearly, Cantelli's bound is still smaller than modified Chebyshev's, for all $b \in (0, \infty)$.

Observe that the event $\{|X| \geq b\}$ under the probability sign in (3) means that X takes on a value outside the symmetric interval $(-b, b)$, whereas the event $\{X \geq b\}$ under the probability sign in (2) means that X takes on a value outside the utterly asymmetric interval $(-\infty, b)$. More generally, one may ask about the exact upper bound on $\mathbf{P}(X \notin (-a, b))$, for any given interval $(-a, b)$ containing 0. The need for such a bound, which would in this sense interpolate between Chebyshev's and Cantelli's, arises naturally in studies of the distributions of the so-called self-normalized sums [4], where one needs a good upper bound on that probability that a quadratic polynomial $X^2 + AX + B$ in a r.v. X will take on a nonnegative value.

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When considering the probability $\mathbf{P}(X \notin (-a, b))$, without loss of generality one may assume that $a \geq b$. Indeed, $\mathbf{P}(X \notin (-a, b)) = \mathbf{P}(-X \notin (-b, a))$, and the r.v. $-X$ is zero-mean and unit-variance whenever X is so. Accordingly, let us present

Theorem 1. *Take any a and b such that $0 < b \leq a < \infty$. Then*

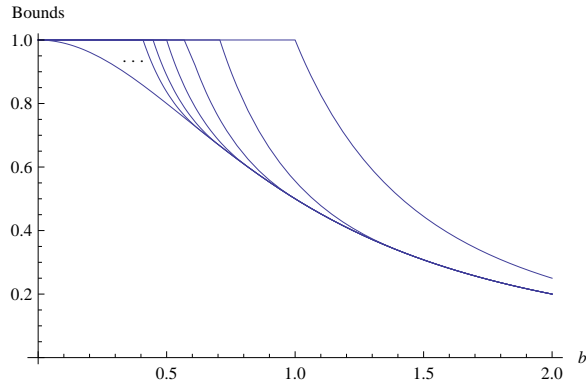
$$\mathbf{P}(X \notin (-a, b)) \leq P_{a,b} := \begin{cases} 1 & \text{if } ab \leq 1, \\ \frac{4 + (a-b)^2}{(a+b)^2} & \text{if } \frac{(a-b)b}{2} \leq 1 \leq ab, \\ \frac{1}{1+b^2} & \text{if } 1 \leq \frac{(a-b)b}{2}, \end{cases} \quad (4)$$

and this upper bound is exact. Moreover,

$$\mathbf{P}(X \notin (-a, b)) \leq 1 \wedge \frac{4 + (a-b)^2}{(a+b)^2} \quad (5)$$

in all of the three cases in (4).

Note that $P_{b,b}$ coincides with the modified Chebyshev bound (3), whereas $P_{\infty,b} := \lim_{a \rightarrow \infty} P_{a,b}$ coincides with the Cantelli bound (2). So, letting $a = kb$ and varying k from 1 to ∞ , one obtains a decreasing family $(P_{kb,b} : 1 \leq k \leq \infty)$ of exact upper bounds interpolating between modified Chebyshev's and Cantelli's. The members of this family of bounds with $k \in \{1, \dots, 6, \infty\}$ are shown in the picture here. One can see that even for such moderate values of the “asymmetry parameter” k as 2 or 3, the improvement of the bound $P_{kb,b}$ over Chebyshev's may be quite significant; for instance, Chebyshev's bound $P_{1,1} = 1$ is 80% greater than $P_{2,1} = \frac{5}{9}$, and it is 100% greater than $P_{k,1} = \frac{1}{2}$ for $k \geq 3$.



Theorem 1 can be proved by a method going back to Chebyshev and Markov; cf. e.g. [3, 1, 2, 5]. Rewrite $\mathbf{P}(X \notin (-a, b))$ as $\int g(X) d\mathbf{P}$, where $g := \chi_{\mathbb{R} \setminus (-a, b)}$. Then one can try to find the best possible upper bound on $\int g(X) d\mathbf{P}$ as $\inf \int f(X) d\mathbf{P}$, where the infimum is taken over all functions f that majorize g and are linear combinations of the moment functions $x \mapsto 1$, $x \mapsto x$, and $x \mapsto x^2$, corresponding to the restrictions $\int d\mathbf{P} = 1$, $\int X d\mathbf{P} = 0$, and $\int X^2 d\mathbf{P} = 1$. So, in our optimization problem the function f is a quadratic polynomial such

that $f \geq g$ on \mathbb{R} . Take now the majorizing $f(x)$ to be $\equiv 1$, $\equiv \left(\frac{2x+a-b}{a+b}\right)^2$, or $\equiv \left(\frac{bx+1}{b^2+1}\right)^2$ in the three respective cases in (4); actually, in all of the three cases one has $g(x) \leq 1 \wedge \left(\frac{2x+a-b}{a+b}\right)^2$ for all $x \in \mathbb{R}$. Next, writing $\mathbf{P}(X \notin (-a, b)) = \int g(X) d\mathbf{P} \leq \int f(X) d\mathbf{P}$, and then taking into account the restrictions $\int d\mathbf{P} = 1$, $\int X d\mathbf{P} = 0$, and $\int X^2 d\mathbf{P} = 1$, one obtains the inequalities in (4) and (5). The exactness of the bound $P_{a,b}$ follows since it is attained when a r.v. X takes on the values

- (i) $-\sqrt{\frac{a}{b}}$ and $\sqrt{\frac{b}{a}}$ with respective probabilities $\frac{b}{a+b}$ and $\frac{a}{a+b}$ — when $ab \leq 1$;
- (ii) $-a$, $\frac{b-a}{2}$, and b with respective probabilities $\frac{2-(a-b)b}{(a+b)^2}$, $\frac{4(ab-1)}{(a+b)^2}$, and $\frac{2+(a-b)a}{(a+b)^2}$ — when $\frac{(a-b)b}{2} \leq 1 \leq ab$;
- (iii) $-\frac{1}{b}$ and b with respective probabilities $\frac{b^2}{1+b^2}$ and $\frac{1}{1+b^2}$ — when $1 \leq \frac{(a-b)b}{2}$.

References

- [1] KARLIN, S., AND STUDDEN, W. J. *Tchebycheff systems: With applications in analysis and statistics*. Pure and Applied Mathematics, Vol. XV. Interscience Publishers John Wiley & Sons, New York-London-Sydney, 1966.
- [2] KEMPERMAN, J. H. B. On the role of duality in the theory of moments. In *Semi-infinite programming and applications (Austin, Tex., 1981)*, vol. 215 of *Lecture Notes in Econom. and Math. Systems*. Springer, Berlin, 1983, pp. 63–92.
- [3] KREĬN, M. G., AND NUDEL'MAN, A. A. *The Markov moment problem and extremal problems*. American Mathematical Society, Providence, R.I., 1977. Ideas and problems of P. L. Čebyšev and A. A. Markov and their further development, Translated from the Russian by D. Louvish, Translations of Mathematical Monographs, Vol. 50.
- [4] PINELIS, I. On the Berry–Esseen bound for self-normalized sums (in preparation).
- [5] PINELIS, I. Optimal tail comparison based on comparison of moments. In *High dimensional probability (Oberwolfach, 1996)*, vol. 43 of *Progr. Probab.* Birkhäuser, Basel, 1998, pp. 297–314.

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